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Potentials and flows associated with a line segment

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SUMMARY
Integral transformations are developed to construct three and five axisymmetric potentials for a needle or straight line segment. These potentials are applied to flow past a needle and one of the main results is for streaming flow past a line segment in which the fluid velocity vanishes on the boundary. This solution may also be regarded as a Stokes flow or an inviscid potential flow.

1. Introduction

The first part of this paper is concerned with the axisymmetric potentials for a needle or straight line segment in three and five dimensions. The needle is not considered as the limiting form of a prolate ellipsoid of revolution or any other closed three-dimensional surface. The method of solution is to take a frame $O(x, y, z)$ and to consider two axially symmetric potentials one about the x-axis and the other about the z-axis. Since both potentials are solutions of the full three-dimensional Laplace equation, they are related by an integral transformation. The main property of the integral transformation is that it maps the mixed value problem for the potential of a thin circular disk into the mixed problem for a finite line segment maintained at the same potential. The method is readily extended to any number of dimensions and in the present work the cases $n=3, 5$ are considered in some detail. The potentials for these cases are continuous but at the tips of the needle there are singularities in the partial derivatives behaving like inverse square root of distance from the tips. This type of singularity is quite common for problems of mixed type. At infinity, the leading term for the potential behaves like a point charge or a source.

The second part of the paper is the application of the potentials to flow past a needle using the correspondence principle of Weinstein [1] and the method of Payne and Pell [2, 3] for a class of axisymmetric inviscid and Stokes flow. The main result is the stream function for streaming flow past a needle satisfying zero fluid velocity on the needle and a uniform stream at infinity. The solution is in fact, an exact solution for potential flow, Stokes flow and of the Navier–Stokes equations. The solution has a continuous velocity, is independent of the Reynolds number and does not possess any vorticity. There is of course no drag on the needle and at infinity the flow behaves like an ordinary fluid producing dipole. It is possible the solution found in this paper has application to slender body theory particularly when the boundary surface is not smooth *e.g.* a finite cylinder of circular cross section with flat ends. In fact, the needle solution may well be the limit of the finite cylinder problem when the radius tends to zero.

Finally, it is shown in the final section there is a non uniqueness for both inviscid and Stokes flow past a needle. However, if the principle of minimum singularity is employed, this non uniqueness can be resolved by choosing the solution least singular at the tips. The solution found here appears to be the only example of a potential flow which satisfies viscous no-slip boundary conditions. This is presumably connected with the fact that the boundary is one-dimensional.

2. Notation and method of solution

It is convenient to define two systems of spherical coordinates referred to the z- and x-axes respectively

$$\begin{aligned} x &= r \sin\theta \cos\phi = r\cos\theta' = z', \\ y &= r\sin\theta\sin\phi = r\sin\theta'\sin\phi' = y', \\ z &= r\cos\theta \qquad = r\sin\theta'\cos\phi' = x', \end{aligned} \tag{1}$$

and $\rho = r\sin\theta$, $\rho' = r\sin\theta'$. The full three-dimensional Laplace operator is defined by

$$\begin{aligned} \Delta_3 &\equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ &\equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \end{aligned} \tag{2}$$

and in (r, θ', φ') coordinates

$$\begin{aligned} \Delta_3' &\equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\right) + \frac{1}{r^2\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} \\ &\equiv \frac{\partial^2}{\partial\rho'^2} + \frac{1}{\rho'}\frac{\partial}{\partial\rho'} + \frac{\partial^2}{\partial z'^2} + \frac{1}{\rho'^2}\frac{\partial^2}{\partial\phi'^2}. \end{aligned} \tag{3}$$

The axially symmetric Laplacian is denoted by

$$L_1 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right), \tag{4}$$

and

$$L_1' \equiv \frac{\partial}{\partial\rho'^2} + \frac{1}{\rho'}\frac{\partial}{\partial\rho'} + \frac{\partial^2}{\partial z'^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\right). \tag{5}$$

It is stated at the outset that the functions considered in this paper are continuous with their partial derivatives up to the order of the differential equations, unless otherwise indicated. Let $V(x, \rho')$ be an axially symmetric harmonic satisfying $L_1'\{V(x. \rho')\} = 0$ and be symmetric with respect to both coordinates x and ρ'. Now V is also a solution of the full Laplace equation

$$\Delta_3(V) \equiv V_{xx} + V_{yy} + V_{zz} = 0, \tag{6}$$

which is symmetric with respect to all three coordinates (x, y, z) and may be expanded in the form

$$V(x, \rho') = \sum_{n=0}^{\infty} V_{2n}(z, \rho)\cos 2n\phi, \tag{7}$$

where $V_{2n}(x, \rho)$ is a solution of the equation

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{4n^2}{\rho^2} + \frac{\partial^2}{\partial z^2}\right) V_{2n}(z, \rho) = 0. \tag{8}$$

It follows from Eqn. (7) that $V_{2n}(z, \rho)$ is expressible in integral form as follows:

$$V_{2n}(z, \rho) = \frac{1}{\pi}\int_0^{2\pi} V\{\rho\cos\varphi, (\rho^2\sin^2\phi + z^2)^{\frac{1}{2}}\}\cos n\phi \, d\phi, \tag{9}$$

for $n \geqq 1$ and

$$W(z, \rho) = V_0(z, \rho) = \frac{1}{2\pi}\int_0^{2\pi} V\{\rho\cos\phi, (\rho^2\sin^2\phi + z^2)^{\frac{1}{2}}\} d\phi, \tag{10}$$

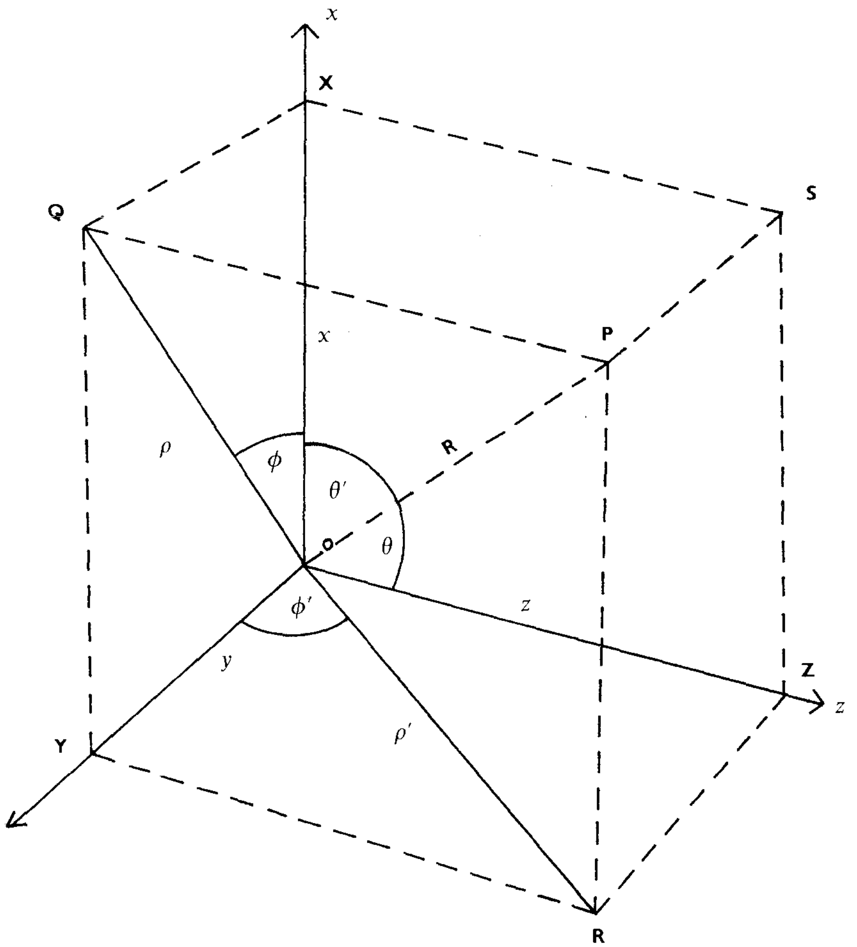

Figure 1. Diagram showing the two systems of spherical polar coordinates (r, θ, ϕ) and (r, θ', ϕ').

where $\rho' = (\rho^2 \sin^2 \phi + z^2)^{\frac{1}{2}}$. Equation (10) is an integral transformation which maps an axially symmetric harmonic $V(z', \rho')$ into an axially symmetric harmonic $W(z, \rho)$. Two properties of the transformation (10) required for the present paper are

$$\text{(i)}\quad W(z, 0) = \frac{1}{2\pi} \int_0^{2\pi} V(0, \rho')\mathrm{d}\phi = V(0, \rho'),\ \rho' = |z|\,, \tag{11}$$

and

$$\text{(ii)}\quad \left.\frac{\partial W}{\partial \rho}\right|_{\rho=0} = \frac{2}{\pi} \left.\frac{\partial V}{\partial x}\right|_{x=0+} . \tag{12}$$

The present paper demonstrates that Eqn. (10) is suitable for mapping the mixed boundary value problem for the potential of a thin circular disk into the mixed problem for the potential of a finite straight line segment.

Now the mixed boundary value problem for the circular disk contains a discontinuity in the normal derivative on the disk surface. In fact

$$L_1(V) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \rho'^2} + \frac{1}{\rho'}\frac{\partial}{\partial \rho'}\right) V(x, \rho') = 2\left.\frac{\partial V}{\partial x}\right|_{x=0+} \delta(x)\,, \tag{13}$$

where $\delta(x)$ is the Dirac delta function defined over the half interval by $\int_0^\infty \delta(x)\mathrm{d}x = \frac{1}{2}$, $\delta(x) = 0$, $x > 0$. It follows that $\partial V/\partial \phi$ is discontinuous at $\phi = \frac{1}{2}\pi, \frac{3}{2}\pi$ for $0 \leqslant r \leqslant 1$, and to show $W(z, \rho)$ is an axially symmetric harmonic in $\rho > 0$, it is first observed that

$$L_1(W) = \int_0^{2\pi} \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right) V(\rho\cos\phi, (\rho^2\sin^2\phi+z^2)^{\frac{1}{2}} d\phi \tag{14}$$

$$= \int_0^{2\pi} \Delta_3(V)\mathrm{d}\phi - \frac{1}{\rho^2}\int_0^{2\pi} \frac{\partial^2 V}{\partial\phi^2}\mathrm{d}\phi\,. \tag{15}$$

But

$$\int_0^{2\pi} \Delta_3(V)d\phi = \int_{\frac{1}{2}\pi-0}^{\frac{1}{2}\pi+0} \Delta_3(V)d\phi + \int_{\frac{3}{2}\pi-0}^{\frac{3}{2}\pi+0} \Delta_3(V)d\phi$$

$$= \frac{1}{\rho^2}\left[\frac{\partial V}{\partial\phi}\right]_{\frac{1}{2}\pi-0}^{\frac{1}{2}\pi+0} + \frac{1}{\rho^2}\left[\frac{\partial V}{\partial\phi}\right]_{\frac{3}{2}\pi-0}^{\frac{3}{2}\pi+0} \tag{16}$$

Since $\Delta_3(V)=\Delta_3'(V)=0$, except on $x=0$, $0\leqslant r\leqslant 1$, and

$$\frac{1}{\rho^2}\int_0^{2\pi} \frac{\partial^2 V}{\partial\phi^2} d\phi = \frac{1}{\rho^2}\left[\frac{\partial V}{\partial\phi}\right]_{\frac{1}{2}\pi-0}^{\frac{1}{2}\pi+0} + \frac{1}{\rho^2}\left[\frac{\partial V}{\partial\phi}\right]_{\frac{3}{2}\pi-0}^{\frac{3}{2}\pi+0}. \tag{17}$$

Combining Eqns. (16) and (17) with Eqn. (15) is follows that W is a solution of $L_1(W)$ in $\rho>0$.

The boundary value problem for the thin circular disk $x=0$, $0\leqslant\rho'\leqslant 1$, is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\rho'^2} + \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\right) V=0\,, \tag{18}$$

except on $x=0$, $0\leqslant\rho'\leqslant 1$, subject to

$$\begin{aligned} &V(0,\rho')=1\,, \quad 0\leqslant\rho'\leqslant 1\,,\\ &\left.\frac{\partial V}{\partial x}\right|_{x=0}=0\,, \quad \rho'>1\,,\\ &V\sim O\left(\frac{1}{r}\right) \text{ as } r\to\infty\,. \end{aligned} \tag{19}$$

One of the simplest representations for the solution is given by Tranter [4] where the disk is regarded as the limiting form of the oblate spheroid and

$$V = \frac{2}{\pi}\cot^{-1}(\sinh\alpha)\,, \tag{20}$$

where oblate ellipsoidal coordinates are defined by

$$x=\sinh\alpha\cos\beta,\quad \rho'=\cosh\alpha\sin\beta\,. \tag{21}$$

In terms of the original coordinates

$$V = \frac{2}{\pi}\cot^{-1}\left\{\frac{r^2-1}{2} + \frac{1}{2}[(r^2-1)^2+4x^2]^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \tag{22}$$

so that W can be expressed as

$$W(z,0) = \frac{1}{\pi^2}\int_0^{2\pi} \cot^{-1}\left\{\frac{r^2-1}{2} + \tfrac{1}{2}[(r^2-1)^2+4x^2]^{\frac{1}{2}}\right\}^{\frac{1}{2}} d\phi\,. \tag{23}$$

It is readily verified that as $r\to\infty$, $W\sim O(r^{-1})$, and on the axis

$$\begin{aligned} W(z,0) &= \frac{2}{\pi}\cot^{-1}\left\{\frac{r^2-1}{1} + \tfrac{1}{2}|r^2-1|\right\}^{\frac{1}{2}},\\ &= 1, \quad 0\leqslant r\leqslant 1\,,\\ &= \frac{2}{\pi}\cot^{-1}(r^2-1)^{\frac{1}{2}} \text{ for } r>1\,. \end{aligned} \tag{24}$$

The potential is continuous on the axis but the derivatives have square root singularities at the tips. To determine the local expansion of the potential W about one tip, say, $\rho=0$, $z=1$, it is convenient to introduce local polar coordinates (r_1, θ_1) defined by

$$r \cos \theta = r_1 \cos \theta_1 + 1, \quad r \sin \theta = r_1 \sin \theta_1 .$$

If $r_1 \ll 1$, and $0 \leqslant \theta_1 < \alpha < \pi$, for some $\alpha < \pi$, then W is approximately given by

$$W^* \sim 1 - \frac{r_1^{\frac{1}{2}}}{\pi} \int_0^{2\pi} \{\cos \theta_1 + [\cos^2 \theta_1 + \sin^2 \theta_1 \cos^2 \phi]^{\frac{1}{2}}\}^{\frac{1}{2}} d\phi . \tag{25}$$

The second term is a constant multiple of $r_1^{\frac{1}{2}} P_{\frac{1}{2}}(\cos \theta_1)$ which is clearly a solution of the three-dimensional Laplace equation and $\partial W^*/\partial \theta_1 = 0$, on $\theta_1 = 0$.

Returning to Eqn. (23) the potential W is continuous everywhere and is a solution of $L_1(W) = 0$ except on $\rho=0$, $|z| < 1$, W satisfies the mixed conditions

$$W(z, 0) = 1, \ |z| < 1, \frac{\partial W}{\partial \rho} = 0, \ \rho=0, \ |z| > 1 , \tag{26}$$

$$W \sim 0 \left(\frac{1}{r}\right) \ \text{as} \ r \to \infty . \tag{27}$$

It is noted that $\partial W/\partial \rho \neq 0$ on the needle but on the other hand it is not infinite except at the tips $\rho=0$, $|z| = 1$. In the next section it will be shown how to extend the method to higher dimensions.

3. Five dimensions

As in the previous section it is convenient to define hyperspherical coordinates:

$$\begin{aligned}
x_1 &= r \sin \theta \sin \phi_1 \sin \phi_2 \cos \phi_3 = = r \cos \theta' = x_5' , \\
x_2 &= r \sin \theta \sin \phi_1 \sin \phi_2 \sin \phi_3 = r \sin \theta' \cos \phi_1' = x_4' , \\
x_3 &= r \sin \theta \sin \phi_1 \cos \phi_2 = r \sin \theta' \sin \phi_1' \cos \phi_2' = x_3' , \\
x_4 &= r \sin \theta \cos \phi_1 = r \sin \theta' \sin \phi_1' \sin \phi_2' \sin \phi_3' = x_2' , \\
x_5 &= r \cos \theta = r \sin \theta' \sin \phi_1' \sin \phi_2' \cos \phi_3' = x_1' ,
\end{aligned} \tag{28}$$

and hypercylindrical radii vectors by

$$\rho = \left\{ \sum_{j=1}^{4} x_j^2 \right\}^{\frac{1}{2}} = r \sin \theta, \ \rho' = \left\{ \sum_{j=1}^{4} x_j'^2 \right\}^{\frac{1}{2}} = r \sin \theta' . \tag{29}$$

Let $V(x_1, \ldots, x_5)$ be a solution of the five-dimensional Laplace equation

$$\Delta_5(V) \equiv \sum_{j=1}^{5} V_{x_j x_j} = 0 , \tag{30}$$

where V is symmetric with respect to each axis x_j, $j=1, \ldots, 5$. It is noted that the full Laplacian in five dimensions can be expressed in cylndrical polars as

$$\Delta_5 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial x_5^2} + \frac{1}{\rho^2 \sin^2 \phi_1} \frac{\partial}{\partial \phi_1} \left(\sin^2 \phi_1 \frac{\partial}{\partial \phi_1} \right) + \frac{1}{\rho^2 \sin^2 \phi_1 \sin \phi_2} \frac{\partial}{\partial \phi_2} \left(\sin \phi_2 \frac{\partial}{\partial \phi_2} \right) + \frac{1}{\rho^2 \sin^2 \phi_1 \sin^2 \phi_2} \frac{\partial^2}{\partial \phi_3^2} .$$

Consider now the analogue of the integral transformation (10) in five dimensions. This is

$$W(x_5, \rho) = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} V(x_1, \ldots, x_5) \sin^2 \phi_1 \sin \phi_2 d\phi_1 d\phi_2 d\phi_3 ,$$

and

$$
\begin{aligned}
L_3(W) &\equiv \left(\frac{\partial^2}{\partial x_5^2}+\frac{\partial^2}{\partial \rho^2}+\frac{3}{\rho}\frac{\partial}{\partial \rho}\right) W \\
&= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi\int_0^\pi \Delta_5(V)\sin^2\phi_1 \sin\phi_2 d\phi_1 d\phi_2 d\phi_3 \\
&- \frac{1}{(\pi\rho)^2}\int_0^\pi\int_0^\pi\int_0^\pi \left\{\frac{1}{\sin^2\phi_1}\frac{\partial}{\partial\phi_1}\left(\sin^2\phi_1\frac{\partial V}{\partial \phi_1}\right)+\right. \\
&\left.\frac{1}{\sin^2\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_2}\left((\sin\phi_2\frac{\partial}{\partial\phi_2}\right)+\frac{1}{\sin^2\phi_1\sin^2\phi_2}\frac{\partial^2 V}{\partial\phi_3^2}\right\}\times \\
&\times \sin^2\phi_1\sin\phi_2 d\phi_1 d\phi_2 d\phi_3
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi\int_0^\pi \Delta_5(V)\sin^2\phi_1\sin\phi_2 d\phi_1 d\phi_2 d\phi_3 \\
&- \frac{1}{(\pi\rho)^2}\int_0^\pi\int_0^\pi\left[\sin^2\phi_1\frac{\partial V}{\partial\phi_1}\right]_0^\pi \sin\phi_2 d\phi_2 d\phi_3 \\
&- \frac{1}{(\pi\rho)^2}\int_0^\pi\int_0^\pi\left[\sin\phi_2\frac{\partial V}{\partial\phi_2}\right]_0^\pi d\phi_1 d\phi_2 \\
&- \frac{1}{(\pi\rho)^2}\int_0^\pi\int_0^\pi\frac{1}{\sin\phi_2}\left[\frac{\partial V}{\partial\phi_3}\right]_0^\pi d\phi_1 d\phi_2 d\phi_2 = 0\,.
\end{aligned}
\tag{34}
$$

This result is still valid when the derivative $\partial V/\partial x_5'$ is discontinuous over the plane $x_5'=0$. Thus $W=v^{(3)}(x_5, \rho)$ is a solution of the five-dimensional axially symmetric Laplace equation. As in the three-dimensional case $V(x_1, \ldots, x_5)$ can be written as $V\equiv \bar{U}(x_5', \rho')$, where $\bar{U}(x_5', \rho')$ is an axially symmetric harmonic in 5 dimensions. The corresponding properties on the axis

(i) $W(x_5, 0)=\bar{U}(0, \rho'),\ \rho'=|x_5|\,,$ (35)

and

(ii) $\left.\dfrac{\partial W}{\partial \rho}\right|_{\rho=0} = \dfrac{4}{3\pi}\dfrac{\partial \bar{U}}{\partial x_5'}(x_5', \rho')|_{x_5'=0+}\,.$ (36)

for the case in which $\partial\bar{U}/\partial x_5'$ is discontinuous on $x_5'=0$. The problem for $\bar{U}$ is that of a hyperdisk $x_5'=0$, $0\leqslant\rho'\leqslant 1$, charged to unit potential. The boundary value problem may be stated as follows:

$$L_3(\bar{U})=0\,, \tag{37}$$

except on the disk and

$$\bar{U}=1,\ x_5'=0,\quad 0\leqslant\rho'\leqslant 1\,, \tag{39}$$

$$\bar{U}\sim O(r^{-3})\ \text{as}\ r\to\infty\,. \tag{40}$$

Following the previous section the most convenient representation for the solution is to introduce oblate spheroidal coordinates by

$$x_5'+i\rho'=\sinh(\alpha+i\beta)\,, \tag{41}$$

and since $\bar{U}$ does not depend on β, the differential equation for $\bar{U}$ is

$$\frac{d}{d\alpha}\left(\cosh^3\alpha\frac{d\bar{U}}{d\alpha}\right)=0\,. \tag{42}$$

If the region exterior to the disk is $\alpha\geqslant 0$, the solution for $\bar{U}$ is readily found to be

$$\bar{U}=\frac{2}{3\pi}\left\{\frac{\sinh\alpha}{\cosh^2\alpha}+3\cot^{-1}(\sinh\alpha)\right\}\,. \tag{43}$$

In terms of the original coordinates. $\bar{U}$ can be expressed as

$$\overline{U} = \frac{2}{3\pi} \frac{\left[\frac{r^2-1}{2} + \frac{1}{2}\{(r^2-1)^2+4\rho^2\sin^2\phi_1\sin^2\phi_2\cos^2\phi_3\}^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\left[\frac{r^2+1}{2} + \frac{1}{2}\{(r^2-1)^2+4\rho^2\sin^2\phi_1\sin^2\phi_2\cos^2\phi_3\}^{\frac{1}{2}}\right]} \tag{44}$$

$$+3\cot^{-1}\left\{\frac{r^2-1}{2} + \frac{1}{2}[(r^2-1)^2+4\rho^2\sin^2\phi_1\sin^2\phi_2\cos^2\phi_3]^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

In this case the integral transformation (32) maps the disk problem into a five-dimensional axially symmetric potential for a straight line segment or needle $|x_5| \leqslant 1$, $\rho=0$. The boundary value problem for W is thus

$$L_3(W) = 0\,, \tag{45}$$

except on the needle $\rho=0$, $|x_5|<1$, subject to

$$W(x_5.\ \rho) = 1,\ \ \rho=0,\ |x_5|<1\,,$$

$$\frac{\partial W}{\partial \rho} = 0,\ \ \rho=0,\ \ |x_5|<1\,,$$

$$W \sim O\left(\frac{1}{r}\right) \text{ as } r\to\infty\,.$$

As in the three-dimensional case $\partial W/\partial\rho \neq 0$ on the needle.

On the axis $\rho=0$,

$$W(x_5, 0) = \frac{2}{3\pi}\left\{\frac{\left[\frac{r^2-1}{2} + \frac{1}{2}|r^2-1|\right]^{\frac{1}{2}}}{\left[\frac{r^2-1}{2} + \frac{1}{2}|r^2-1|\right]} + 3\cot^{-1}\left[\frac{r^2-1}{2} + \frac{1}{2}|r^2-1|\right]^{\frac{1}{2}}\right\}, \tag{47}$$

where $|x_5|=r$, when $\rho=0$. It follows that W is continuous at the tips of the needle, but the derivative is infinite like $\varepsilon^{-\frac{1}{2}}$, where ε is the distance from a tip. If x_5 is replaced by z, so that $r=(\rho^2+z^2)^{\frac{1}{2}}$, then

$$W(z, \rho) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi\int_0^\pi \overline{U}\sin^2\phi_1\sin\phi_2\,d\phi_1\,d\phi_2\,d\phi_3 \tag{48}$$

where $\overline{U}$ is defined by Eqn. (44).

4. Application to streaming flow

The fluid velocity $\boldsymbol{q}$ for axially symmetric incompressible viscous flow can be expressed in terms of a stream function $\psi(z, \rho)$ by

$$\boldsymbol{q} = \text{curl}\left\{-\frac{\psi}{\rho}\hat{\phi}\right\}, \tag{49}$$

where ψ satisfies the vorticity equation

$$L^2_{-1}(\psi) = R\rho\,\frac{\partial(\psi, l)}{\partial(z, \rho)},\quad L_{-1} \equiv \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}, \tag{50}$$

where $l=L_{-1}(\psi)/\rho^2$ is the ring vorticity and R is the Reynolds number for the flow. If the motion is that of streaming flow past a fixed obstacle R can be defined as $R=\overline{U}_\infty a/\nu$, where $\overline{U}_\infty$ is the speed of the stream at infinity, a is a typical length scale for the obstacle and ν is the kinematic viscosity. A known particular solution of Eqn. (50) is the potential flow

$$L_{-1}(\psi) = 0\,. \tag{51}$$

Solutions of Eqn. (51) are in general not useful to viscous flow because no slip boundary conditions cannot be satisfied on a fixed surface. However, in the case of a line segment it will be shown that a solution can be constructed for streaming flow past a needle in which the fluid velocity vanishes on the part of the axis occupied by the needle. First it is noted that the solution satisfying the inviscid boundary condition on the needle is just $\psi=\frac{1}{2}\rho^2$, so that the fluid does not recognize the presence of the boundary. If the fluid is viscous the boundary value problem for the needle will be defined as follows: ψ is a solution of $L_{-1}(\psi)=0$ except on $\rho=0$, $|z|\leqslant 1$, satisfying

$$\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=\frac{1}{z}\frac{\partial\psi}{\partial\rho}=0,\quad \rho=0,\ |z|<1\,,$$

$$\psi=L_{-1}(\psi)=0,\qquad \rho=0,\ |z|>1\,, \tag{52}$$

and

$$\psi\sim\tfrac{1}{2}\rho^2 \text{ as } r\to\infty\,.$$

To solve this problem it is convenient to employ the Weinstein correspondence principle and write ψ in the form

$$\psi=\tfrac{1}{2}\rho^2\{1-v^{(3)}(z,\rho)]\,, \tag{53}$$

where $v^{(3)}(z,\rho)$ is an axially symmetric harmonic in 5 dimensions satisfying the conditions

$$v^{(3)}(z,0)=1,\quad |z|<1\,,$$

$$\frac{\partial v^{(3)}}{\partial\rho}=0,\quad \rho=0,\ |z|>1\,, \tag{54}$$

and

$$v^{(3)}\sim O(r^{-3}) \text{ as } r\to\infty\,.$$

This is the problem solved in the previous section and can be expressed as

$$v^{(3)}(z,\rho)=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\int_0^\pi \overline{U}\sin^2\phi_1\sin\phi_2\,d\phi_1\,d\phi_2\,d\phi_3\,, \tag{55}$$

where $\overline{U}$ is defined by Eqn. (44). It is readily checked that the fluid velocity vanishes on $\rho=0$, $|z|\leqslant 1$, and behaves like a uniform stream at infinity. Since $v^{(3)}\sim O(r^{-3})$. It follows that the leading order terms at infinity consist of a uniform stream and a fluid producing dipole. The fluid velocity is continuous near the tips of the needle but the velocity derivatives contain weak singularities in these regions. This follows from Eqn. (47) since the velocity on the axis is

$$u(z,0)=-\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right)_{\rho=0}=v^{(3)}(z,0)-1\,, \tag{56}$$

$$=\frac{2}{3\pi}\left\{\frac{\left[\frac{z^2-1}{2}+\frac{1}{2}|z^2-1|\right]^{\frac{1}{2}}}{\left[\frac{z^2+1}{2}+\frac{1}{2}|z^2-1|\right]}+3\cot^{-1}\left[\frac{z^2-1}{2}+\frac{1}{2}|z^2-1|\right]^{\frac{1}{2}}\right\}-1\,. \tag{57}$$

To obtain the leading terms in the expansion of $W(z,\rho)$ about a tip introduce local polar coordinates by

$$r\cos\theta=1+r_1\cos\theta_1,\ r\sin\theta=r_1\sin\theta_1\,,$$

then from Eqn. (48) it is readily shown that

$$W\sim 1-\frac{4}{3\pi^3}r_1^{\frac{1}{2}}\int_0^\pi\int_0^\pi\int_0^\pi\{\cos\theta_1+$$

$$[\cos^2\theta_1+\sin^2\theta_1\sin^2\phi_1\sin^2\phi_2\cos^2\phi_3]^{\frac{1}{2}}\}^{\frac{1}{2}}\sin^2\phi_1\sin\phi_2\,d\phi_1\,d\phi_2\,d\phi_3\,, \tag{58}$$

for $0 \leqslant \theta_1 < \alpha < \pi$.

The solution of Eqn. (54) is also a solution of the inviscid flow problem but the trivial solution $\psi = \frac{1}{2}\rho^2$ may be preferred for inviscid flow as it contains no singularity at the tips.

5. Stokes flow

Before considering the Stokes flow past a line segment it is worth considering briefly the corresponding flow past an ellipsoid of revolution and its limit as the surface degenerates into a line joining the foci. Defining prolate ellipsoidal coordinates by

$$z + i\rho = \cosh(\xi + i\eta), \tag{59}$$

the exterior of the ellipsoid is defined by $\xi \geqslant \xi_0$, $0 \leqslant \eta \leqslant 2\pi$. The stream function ψ for flow past an ellipsoid of revolution $\xi \geqslant \xi_0$ has been found by Payne and Pell [3] and can be expressed in the form

$$\psi = \frac{1}{2}\rho^2 \left\{ 1 - \frac{\left[\dfrac{s(s_0^2-1)}{s_0^2+1} - \dfrac{1}{2}(s_0^2+1)\log\dfrac{s+1}{s-1} \right]}{\left[s_0 - \dfrac{1}{2}(s_0^2-1)\log\dfrac{s_0+1}{s_0-1} \right]} \right\}, \tag{60}$$

where $s = \cosh\xi$, $s_0 = \cosh\xi_0$. As $\xi_0 \to 0$, the ellipsoid shrinks to a line joining the foci $z = \pm 1$, $\rho = 0$, and the stream function ψ is given by

$$\psi = \tfrac{1}{2}\rho^2, \tag{61}$$

which is the potential flow solution. However the velocity components are given by

$$u = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} \begin{rcases} = -1, & \alpha > 0 \\ = 0, & \alpha = 0 \end{rcases}, \quad v = \frac{1}{\rho}\frac{\partial\psi}{\partial x} = 0, \tag{62}$$

so that the flow is discontinuous. The drag also tends to zero. Such a flow is unrealistic but it is possible to find a continuous solution using the potentials of the previous sections. First from the Weinstein decomposition formula for iterated operator equations, a solution of the Stokes equation

$$L_{-1}^2(\psi) = 0 \tag{63}$$

can be represented in the form

$$\psi = \tfrac{1}{2}\rho^2[1 - A_1 v^{(1)}(z,\rho) - A_3 v^{(3)}(z,\rho)]. \tag{64}$$

$v^{(k)}(z,\rho)$ is a solution of $L_k v^{(k)}) = 0$, where the operator

$$L_k \equiv \frac{\partial^2}{\partial\rho^2} + \frac{k}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}.$$

Now if $v^{(k)}(z,\rho)$, $k = 1, 3$, satisfy the boundary conditions

$$\begin{aligned} & v^{(k)}(z, 0) = 1, \ |z| < 1, \\ & \frac{\partial v^{(k)}}{\partial\rho} = 0, \ \rho = 0, |z| > 1, \\ & v^{(k)} \sim O(r^{-k}) \ \text{as} \ r \to \infty, \end{aligned} \tag{65}$$

then the velocity vanishes on the needle and satisfies the outer boundary condition if $A_1 + A_3 = 1$. The flow is not uniquely determined and $v^{(1)}$, $v^{(3)}$ are defined by the expressions (23) and (48). The solution which is least singular at the tips corresponds to setting $A_1 = 0$. This implies there is no vorticity in the fluid and hence the line segment experiences no drag. The pressure is also finite at the tips. If $A_1 \neq 0$, there is a finite non-zero drag on the line segment and the pressure is infinite like and inverse square root of distance from the tips.

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